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Note

Pseudo-spectral Simulation of a Two-Dimensional Vortex Flow in a Stratified, Incompressible Fluid

1. Introduction

In recent years, there has been growing interest in spectral methods and their usefulness in calculating all types of flows. Problems which have been attacked by spectral methods include transition in plane Poiseuille and Couette flow [1], barotropic motions of the ocean [2], and atmospheric diffusion [3], to name only a few. Work by two of the present authors has dealt with solutions to the Navier–Stokes equations for incompressible flow [4–6]. In this paper, the authors report experience in computing inviscid, stratified, two-dimensional flows.

2. Statement of the Problem

The authors investigated the motion of a two-dimensional vortex of finite radius in a corner of a semi-infinite medium with solid boundaries at $x=0$ and $y=0$, as shown in Fig. 1. The computational domain is bounded by the lines $x=0$, $x=L$, $y=0$, and $y=-\mathrm{H}$. The solid boundaries can be viewed as the result of image vortices in the flow placed outside the computational domain in such a way as to generate a $\psi=0$ streamline along the lines $x=0$ and $y=0$. For a line vortex of radius much smaller than the distance to the nearest boundary, a procedure for solving this problem exactly is well-known [7] and serves as a guide to the numerical study. The initial vorticity distribution in an infinite medium is assumed to be of the form

$$\omega(x,y)=\frac{\Gamma}{2\pi r_0}\exp\{[-(x-x_0)^2-(y-y_0)^2]/2r_0^2\},$$

where Γ is the circulation and r_0 is the nominal vortex core radius. In the presence of solid boundaries, "image" vorticity of appropriate sign and location is added to ensure $\omega=0$ at such boundaries. In the absence of stratification, one expects such a vortex to move in much the same way as a line vortex. Stratification may, depending upon the sign of the vorticity, either aid or oppose this motion but will, in any case, certainly distort the initial vorticity distribution. The density was taken to vary linearly with depth.

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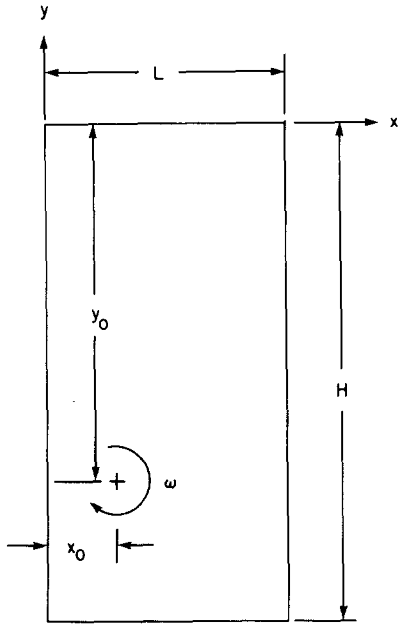

FIG. 1. Schematic representation of the initial vorticity distribution. The computational domain is $0 \leqslant x \leqslant L$, $-H \leqslant y \leqslant 0$.

The equations of motion are, under the Boussinesq approximation,

$$\frac{D\omega}{Dt} = -\frac{g}{\rho}\frac{\partial\rho}{\partial x}, \tag{1}$$

$$\frac{D\rho}{Dt} = 0, \tag{2}$$

$$\nabla^2\psi = \omega, \tag{3}$$

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{4a, b}$$

For the solution of the Poisson equation (3), boundary conditions are required on the stream function at all boundaries. A finite difference code [8] developed for the same problem computes ψ on the boundary from the Biot–Savart integral solution to Poisson's equation. For the nonuniform mesh implied by Chebyshev polynomials, such a computation is not practical; for N modes in x and M modes in y, it would involve either storing $N^2M + M^2N$ values of the logarithmic kernel of the Biot–Savart integral, or recomputing these values at every time step. Imposition of simple boundary conditions, such as $\psi = 0$ or $\partial\psi/\partial n = 0$, is, however, relatively easy, even with Chebyshev polynomials. For $x = 0$ and $y = 0$, the rigid wall condition, $\psi = 0$, is appropriate. For the boundaries at $x = L$ and $y = -H$, imposition of $\psi = 0$ would be equivalent to placing an incorrect vortex image of opposite sign to the actual vortex outside of these boundaries. The parallel flow condition at $x = L$ or $y = -H$; i.e., $\partial\psi/\partial n = 0$, is also not correct; it can be viewed as being produced principally by a vortex image of like sign outside these boundaries. An approximately correct solution may be obtained by solving two separate Poisson problems, one with a rigid wall boundary condition and the other with a parallel flow boundary condition. The Poisson problem being linear, the solutions to each of the problems may be superimposed

$$\psi = \tfrac{1}{2}(\psi_1 + \psi_2).$$

This procedure has the effect of essentially removing the influence of unwanted images; the first false image is cancelled out, leaving a dipole array of false images centered $3L$ or $3H$ from the $x = L$ or $y = -H$ boundary, the effect of which on the solution is negligible. It should be noted that, although the resulting problem is periodic over a distance $4L$ in the x-direction and $4H$ in the y-direction, it is certainly no longer periodic over the computational domain.

For the convected variables, vorticity and density, one could apply antisymmetry and symmetry conditions, respectively, at the rigid boundaries. That is,

$$\omega = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad y = 0,$$

$$\frac{\partial\rho}{\partial n} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad y = 0.$$

Such boundary conditions are required in finite difference simulations but, since no flow crosses these boundaries, it was anticipated these boundary conditions would be unnecessary for the pseudo-spectral simulation, and this was, in fact, the case. It has been argued by Charney *et al.* [9] that solution of the 2-D vorticity equation (1) would require specification of the vorticity only at inflow points. Since there is no *a priori* knowledge of vorticity or density being convected from outside the computational domain, inflow convection for the corner flow case was simply set to zero in the current computations. It is worth noting that when this was not done, the computation was unstable, as expected. No special computation was made at outflow points on the boundary.

3. Numerical Scheme

The solution for the flow was developed by assuming the stream function, vorticity and density can be represented by functions of the form

$$f(x, y, t) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{n,m}(t)\, T_n^*\left(\frac{x}{L}\right) T_m^*\left(\frac{y}{H}\right)$$

in which $a_{n,m}(t)$ are unknown functions of time to be determined from the stream function vorticity equations and T_n^* is the Chebyshev polynomial of order n in the interval $0 \leqslant x/L \leqslant 1$. The values of $a_{n,m}$'s for the vorticity were determined by integrating Eq. (1) in real space by the Adams–Bashforth method and then using a Fast Fourier Transform routine to find the $a_{n,m}$'s from the spatial values of vorticity. In the time integration of the vorticity equation, it is necessary to compute the values of the convective terms at each so-called "natural point" in physical space. These values are tabulated by computing the velocities from the stream function expansions and the vorticity derivatives from the vorticity expansion. Note that these expansions are from the previous time integration since the Adams–Bashforth scheme is explicit. This approach (collocation) [14] for solving for the vorticity equation was adopted over solving the problem in spectral space due to the ease of satisfying boundary conditions and also due to the simplicity. A similar procedure is used to integrate the density equation in time.

The Poisson equation was solved in spectral space by a variation of the tensor-product method [10]. The operator $\partial^2/\partial x^2$ is diagonalized by means of a similarity transformation and the $\partial^2/\partial y^2$ operator is made diagonal by integrating the Poisson equation twice in y [11]. Boundary conditions are applied by means of the "tau" method [13].

4. Time-Step Size

We have chosen to select the time step based on a local Courant number [12],

$$\Delta t = C_{\max} \min_{\substack{0 \leqslant i \leqslant N \\ 0 \leqslant j \leqslant M}} \{\Delta x_i/u(x_i, y_j),\ \Delta y_j/v(x_i y_j)\},$$

where Δx_i, Δy_j are computed from the spacing of the natural points. $C_{\max}$ was typically chosen to be 0.2.

5. Smoothing Algorithm

Straightforward application of the numerical procedure outlined in Section 3 to the problem outlined in Section 2, produced grid-scale ripples in the vorticity distribution which became unacceptably large with time. These ripples were insensitive to the choice of time step and highly resistant to the addition of artificial dissipative terms.

The elimination of this "short wavelength garbage," as Boris and Book [16] have so aptly called it, has been one of the objects of the present study. We hypothesize, as suggested by Boris and Book, that the source of the noise is in the amplification of the inevitable over- and undershoots which occur when a function with a steep (even though finite) gradient is represented by any finite series of terms.

It is easy, of course, to build a dissipative mechanism which reduces or eliminates high-frequency components of a solution; the problem is to do this without also destroying the physically meaningful portion of the solution. For example, one possible strategy is to apply some low-pass filter to the solution either at every time step or, as suggested by Haidvogel *et al.* [2] and independently by Gottlieb and Orszag [15], at intervals of many time steps. The authors have experimented with many such filters, including one of the precise form suggested by Haidvogel *et al.*, with disappointing results. Any filter stiff enough to control the noise also produced an unacceptable smearing of the main vortex.

An entirely different approach is to attempt to use an understanding of the hypothesized source of the noise to control it. When a function, vorticity for example, is expanded in Chebyshev polynomials, the approximating function will pass through specified values at the natural points exactly. In the vicinity of a steep gradient, the approximate function will display alternate maxima and minima between natural points. As the vorticity is convected, usually by less than one full "grid" spacing, these extrema will appear at the natural points. What is characteristic of these extrema is that, when they first appear, they are local—a maximum followed one grid point away by a minimum, and so on. An entire class of schemes, originated by Boris and Book [16, 17], and cumulatively termed "Flux-Corrected Transport," has been created to control such local extrema. The authors attempted to employ such schemes for the current problem with virtually no success—the problem being that the "weak clipping" over three grid points reported by Boris and Book is unacceptable when the phenomenon of interest is at most five grid points wide. A generalization of the original one-dimensional schemes to multiple dimensions [18] shows more promise, but again implementation of the scheme for the problem at hand has produced unsatisfactory results.

The most satisfactory results to date have been obtained by a rather simple-minded approach. In this procedure, the solution is advanced in time by first including x convection, smoothing the solution by an heuristic rule in the x-direction, adding y convection and then smoothing the resulting solution in the y-direction. Note that the solution does not employ the classical splitting approach since the convective terms are both computed from the same previous time-step solutions. The heuristic rule is that, after convection in the x-direction, the solution is examined at the natural points for each constant y. If the solution at a given point is not bounded by its neighbors in x, then the solution is examined over five points. Let x_j be the suspect point, $f(x_j)$ be the solution at that point, and $s_{j+1/2} = f(x_{j+1}) - f(x_j)$, and so on. Then, if

$$s_{j+1/2} \cdot s_{j+3/2} < 0$$

and

$$s_{j-1/2} \cdot s_{j-3/2} < 0,$$

a diffusive correction is added at x_j:

$$\bar{f}(x_j) = f(x_j) + \nu[s_{j+1/2} - s_{j-1/2}],$$

and correspondingly subtracted at neighboring points. In the current study, ν was taken to be 0.1. The procedure is also applied in the y-direction. The most that can be expected of such a scheme is to control the solution at the natural points; the remaining grid-scale noise may be removed by some after-the-fact, "cosmetic" low-pass filtering. It is important to note, however, that no amount of cosmetic filtering can restore solutions computed with the damping coefficient $\nu = 0$ to a similarly smooth state. The "cosmetic" spectral filter was of a form suggested by Haidvogel *et al.* [2]:

$$a_{nm}(t, \text{filtered}) = f_n f_m a_{nm}(t),$$

where

$$f_n = \frac{1.0 - \exp[-(N^2 - n^2)/N_0^2]}{1.0 - \exp[-(N^2/N_0^2)]}.$$

N_0 was typically taken to be $2N$. This filter was chosen since it produced qualitatively satisfactory results; the optimum choice of filter will obviously depend on the needs of a particular problem.

6. Numerical Results

Typical results of the computations are presented in Figs. 2 and 3. For this case, we chose a strong, constant stratification ($\partial \rho / \partial y = \text{constant}$), such that $2Nx_0/v_0$ was

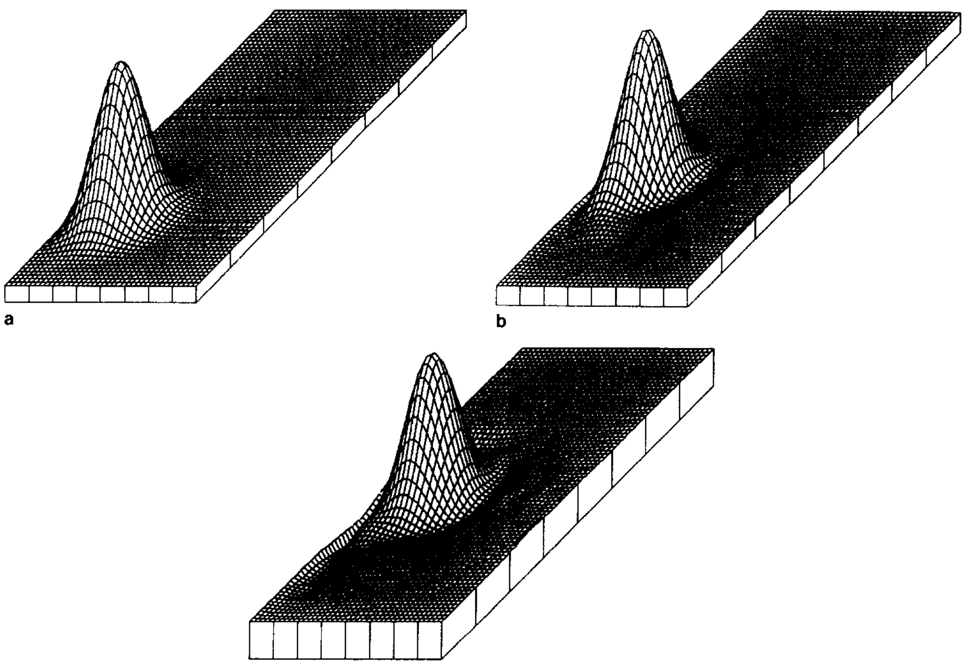

FIG. 2. Perspective plot of vorticity at representative times. The origin of coordinates is in the upper left-hand corner, x increases to the right, and y increases from the bottom to the top of the plot. The mesh which is plotted is much finer than the computational "grid"; 17 modes were used in the x-direction and 33 modes in the y-direction. (a) $Nt = 0$. (b) $Nt = 0.58$. (c) $Nt = 1.35$.

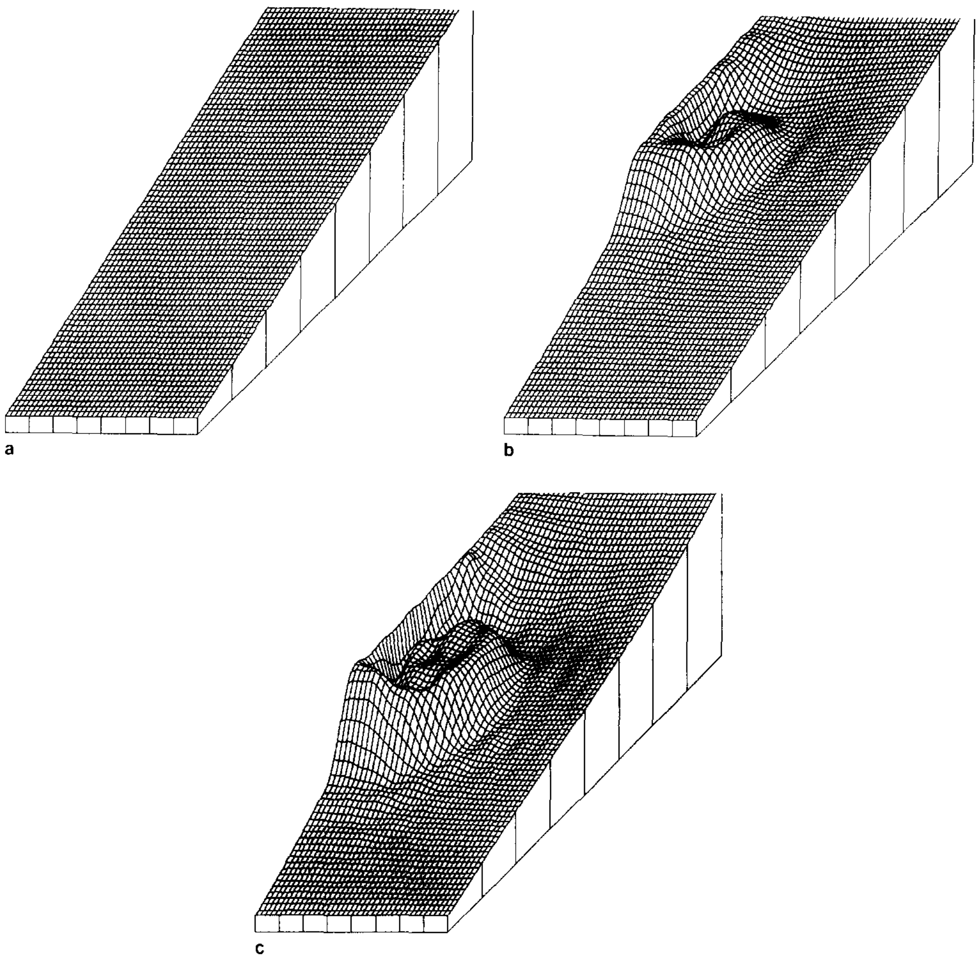

FIG. 3. Perspective plot of density at representative times. For clarity, the origin of coordinates is in the lower left-hand corner, with y increasing from top to bottom and x increasing from left to right. (a) $Nt = 0$. (b) $Nt = 0.58$. (c) $Nt = 1.35$.

equal to 1, where $N = [(-g/\rho)(\partial\rho/\partial y)]^{1/2}$ is the Brunt–Vaisala frequency and $v_0 = \Gamma/2\pi x_0$ is the initial upward velocity of the vortex [7]. Computations made with a finite difference code developed by St-Cyr [8] are in substantial agreement with those presented in Figs. 2 and 3.

One obvious dynamical effect of stratification, as evidenced in Fig. 2, is the production of countersign vorticity around the vortex core. Since the fluid is incompressible, the density plots (Fig. 3) provide a snapshot of the vertical displacement of the fluid from its original position. Note that fluid near the centerline, where the vertical velocity is the greatest, is trailing the main vortex; heavier fluid that would have been convected with the vortex in an unstratified medium is instead "draining" from the vortex at the centerline.

7. Discussion

One may at the outset expect difficulties in computing highly nonlinear inviscid flows with spectral or pseudo-spectral methods. The basic difficulty is that, even for two-dimensional problems, the nonlinear terms will produce an energy cascade from low wavenumbers to high. In the absence of a dissipative mechanism to remove energy from high wavenumbers, a spectrally truncated representation of the Euler equations will whiten with time and eventually become equipartitioned, even if the effects of aliasing are completely eliminated. Nonlinear convective processes for the problem at hand take place on a time scale of the order x_0/v_0; the linear effects of buoyancy, on the other hand, have a time scale of order $1/N$. The parameter v_0/Nx_0 which is the ratio of the buoyancy time scale to the convective time scale, is thus a measure of the nonlinearity of the flow; it has a role similar to the Rossby number ε in simulations of Rossby waves. Since $v_0/Nx_0 = 2$ for the simulation presented here, substantial spectral energy transfer by nonlinear processes may be expected on a time scale $Nt \approx 1$. The difficulties reported by Haidvogel *et al.* [2] for $\varepsilon \approx 1$ are a result of substantial spectral energy transfer taking place over one period of a Rossby wave; the heuristic fix suggested by Haidvogel *et al.* is simply a means of taking the energy out of high wavenumbers which would otherwise collect there. The current simulation is subject to the same problem and the heuristic smoothing mechanism presented here is essentially a nonlinear dissipative mechanism which operates most strongly at the highest wavenumber in the spectrum.

One reviewer has pointed out to the authors that the Charney–Fjortoft–Von Neumann closure used in these simulations may result in an unphysically large vorticity gradient at a point where the flow is tangent to the boundary [19]. The authors observed no such phenomenon in the study reported here; it may be that the smoothing mechanism, which was introduced to alleviate problems which occur even in a completely closed box, prevented such a problem from occurring. Indeed, the dissipation due to the smoothing mechanism presented here is greatest near the boundary, where the mesh is very fine.

This study was originally undertaken to obtain a faster means of solving stratified flow problems than existing finite difference codes. Comparison of computer codes for speed requires a careful study, and it is not our purpose to report such a study here. Preliminary comparisons indicate, however, that the pseudo-spectral code may be as much as 10 times faster than comparable finite difference codes. The experience reported here, however, suggests that the inviscid pseudo-spectral code will be most usefully applied to problems more nearly linear than the one presented here.

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